Math 255A' Lecture 8 Notes

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October 14, 2019

1 Locally Convex Topological Vector Spaces

1.1 Topologies generated by seminorms

Definition 1.1. A **topological vector space (TVS)** over $\mathbb{F}$ is a vector space $(X, \mathcal{T})$ with a topology such that

1. $X \times X \to X$ sending $(x, y) \mapsto x + y$ is continuous,
2. $\mathbb{F} \times X \to X$ sending $(\alpha, x) \mapsto \alpha x$ is continuous.

Let X be a vector space over $\mathbb{F}$, and let $\mathcal{P}$ be a family of seminorms on X. We can use $\mathcal{P}$ to generate a topology (like how we do with norms). We get a base for the topology given by

$$\left\{ \bigcap_{i=1}^{k} \{x : p(x - x_i) < \varepsilon_i\} : p_1, \dots, p_k \in \mathcal{P}, x_1, \dots, x_k \in X, \varepsilon_i, \dots, \varepsilon_k > 0 \right\}.$$

Definition 1.2. A TVS X is a **locally convex space (LCS)** if the topology is is generated by some family $\mathcal{P}$ of seminorms and $\bigcap_{p \in \mathcal{P}} \{x : p(x) = 0\} - \{0\}$ (the seminorms separate points).

Proposition 1.1. *Let X be a TVS, and let p be a seminorm on X. The following are equivelent:*

1. *p is continuous.*
2. *$\{x : p(x) < 1\}$ is open.*
3. *$0 \in \operatorname{int}\{x : p(x) < 1\}$.*
4. *$0 \in \operatorname{int}\{x : p(x) \leq 1\}$.*
5. *p is continuous at 0.*

6. There is a continuous seminorm q such that $p \leq q$.

Proof. The first four statements get weaker, so we have (1) $\implies$ (2) $\implies$ (3) $\implies$ (4).

(4) $\implies$ (5): Let $\varepsilon > 0$. Then

$$U = \operatorname{int}\{x : p(x) \leq \varepsilon/2\} = \varepsilon/2 \cdot (\operatorname{int}(\{x : p(x) \leq 1\})).$$

(5) $\implies$ (1): Compose with translations.

(6) $\implies$ (1): Suppose taht $p \leq q$. Then $0 \in \{q < 1\} \subseteq \operatorname{int}\{p < 1\}$. □

Proposition 1.2. *Let $p_1, \ldots, p_n$ be continuous seminorms. Then $p_1 + \cdots + p_n$ and $\max_i p_i$ are continuous seminorms.*

Proposition 1.3. *If $(p_i)_i$ is a family of continuous seminorms and $p_i \leq q$ for all i, where q is a continuous seminorm, then $\sup_i p_i$ is continuous.*

Example 1.1. Let X be a (Tychonoff)[1] topological space, let $K \subseteq X$ be compact, and let $p_K(f) = \|f|_K\|_{\sup}$. Then $\{p_K : K \subseteq X \text{ compact}\}$ generate a locally convex topology.

On $\mathbb{R}^n$, this topology is generated by $\{p_{\overline{B(0,n)}} : n \in \mathbb{N}\}$.

Example 1.2. Let X be a normed space. For any $f \in X^*$, let $p_f(x) = |f(x)|$. Then X with the resulting LCS structure is called X with the **weak topology**.

1.2 Convex sets

Definition 1.3. Let X be a vector space, and let $A \subseteq X$. The **convex hull** of A is

$$\operatorname{co} A := \bigcap\{C : C \supseteq A, C \text{ convex}\}.$$

The **closed convex hull** of A is

$$\overline{\operatorname{co}} A := \bigcap\{C : C \supseteq A, C \text{ convex and closed}\}.$$

Proposition 1.4. $\overline{\operatorname{co}} A = \overline{\operatorname{co} A}$.

Proof. ($\supseteq$): The left hand side, closed, convex and contains A.

($\subseteq$): It suffices to show that $\overline{\operatorname{co} A}$ is convex. Consider $c = ta + (1-t)b$ for a $a, b \in \overline{\operatorname{co} A}$ and $0 < t < 1$. Consider $F : X \times X \to X$ given by $(x, y) \mapsto tx + (1-t)y$; F is continuous. Then for any neighborhood $W \ni c$, there is a neighborhood $W' \ni (a, b)$ such that $F[W'] \subseteq W$. By the definition of the product topology, we can find a neighborhood $U \times V \subseteq W'$ with the same property. Now pick $a' \in U \cap \operatorname{co} A$ and $b' \in B \cap \operatorname{co} A$. Now $F(a', b') \in W \cap \operatorname{co} A$. So $c \in \overline{\operatorname{co} A}$, as desired. □

[1] This means that it is Hausdorff and whenever $x \imath n X$ and $A \subseteq X$ is closed, there is an $f \in C(X)$ such that $f(x) = 0$ and $f|_A = 1$. If X is not Tychonoff, this still works, but the space is actually very small.

1.3 Correspondence between nice convex sets and seminorms

Definition 1.4. Let X be a vector space over $\mathbb{F}$, and let $A \subseteq X$ be convex.

1. A is **balanced** if $\alpha A \subseteq A$ for all $\alpha \in \mathbb{F}$ and $|\alpha| \leq 1$.
2. A is **absorbing** if for all $x \in X$, there is a $\beta \in (0, \infty)$ such that $x \in \beta A$.
3. A is **absorbing at** $a \in A$ if $A - a$ is absorbing.

Proposition 1.5. *Let X be a vector space over $\mathbb{F}$. If V is a nonempty, balanced, convex set which is absorbing at all its points, then there is a unique seminorm on X such that $V = \{x : p(x) < 1\}$.*

Proof. Define $p(x) := \inf\{t \geq 0 : x \in tV\}$. This is called the **Minkowski functional** of V. Then p is a seminorm:

- (homogeneity): $p(\alpha x) = \inf\{t : \alpha x \in tV\} = |\alpha| \inf\{t : \frac{\alpha}{|\alpha|} x\} = |\alpha| p(x)$.
- (subadditivity): If x, y, suppose $x \in tV$ and $y \in sV$. Then $x + y \in tV + sV = (t+s)V$ (by convexity). So if $p(x) \leq t$ and $p(y) \leq s$, then $p(x+y) \leq t + s$.

If $p(x) < 1$, then $x \in tV$ for some $t < 1$. Because V is balanced, $V \supseteq tV$, so $x \in V$. This gives $\{p < 1\} \subseteq V$.

Conversely, suppose $x \in V$. Then $p(x) \leq 1$. Since V is absorbing at x, there exists some $\varepsilon > 0$ such that $x + \varepsilon x \in V$. So $p(x) \leq 1/(1+\varepsilon) < 1$. This gives $V \subseteq \{p < 1\}$.

Uniqueness: if seminorms satisfy $\{p < 1\} = \{q < 1\}$, then $p = q$ (from lecture 1). □

Corollary 1.1. *A TVS is a LCS if and only if the collection of convex, balanced sets absorbing all their own points is a neighborhood base at 0.*

Proposition 1.6. *A LCS is generated by a translation invariant metric if and only if it is generated by a countable family of seminorms.*

Proof. If $(p_n)_{n=1}^{\infty}$ is a sequence of seminorms, then we can define

$$d(x, y) := \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x-y)}{1 + p_n(x-y)}.$$

□

Definition 1.5. A convex set A in a TVS X is **bounded** if for any neighborhood U of 0, there is a $t < \infty$ such that $tU \supseteq A$.

Theorem 1.1. *A LCS is normable if and only if it has a bounded, open neighborhood of 0.*